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LARGE SAMPLE THEORY FOR A BAYESIAN NONPARAMETRIC SURVIVAL CURVE ESTIMATOR BASED ON CENSORED SAMPLES

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ABSTRACT

Let $X_1,\ldots,X_n$ be iid F and let $Y_1,\ldots,Y_n$ be independent (and independent also of $X_1,\ldots,X_n$) random variables. For i = 1,...,n, set $\delta_i = 1$ if $X_i \le Y_i$ and $= 0$ if $X_i > Y_i$, and $Z_i = \min\{X_i, Y_i\}$. Then assuming that F is distributed according to a Dirichlet process (Ferguson, (1973) Ann. Stat. 2, 209-230) with parameter α, the authors (1976), J. Amer. Stat. Assoc., 61, December) obtained the Bayes estimator $\hat{F}_\alpha$ of F under the loss function $L(F,\hat{F}) = \int (F(u)-\hat{F}(u))^2 dw(u)$ using $(\delta_1,Z_1),\ldots,(\delta_n,Z_n)$. Now let $X_1,\ldots,X_n$ be iid F_0 and $Y_1,\ldots,Y_n$ be iid G where both F_0 and G are unknown continuous distributions. In this paper, it is shown that $\hat{F}_\alpha$ is mean square consistent with rate $O(n^{-1})$ and almost sure consistent with rate $O(\log n/\sqrt{n})$. Also, it is established that $\{\hat{F}_\alpha(u) \mid 0 < u \le T\}$, $T < \infty$, converges weakly to a Gaussian process whose covariance structure coincides with the limiting covariance structure obtained from Kaplan-Meier's ((1958), J.Amer. Stat. Assoc., 53, 457-481) product limit estimator by Breslow and Crowley ((1974), Ann. Stat. 2, 437-453).

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V. Susarla[1,2,3] and J. Van Ryzin[1,2,4]

1. INTRODUCTION AND SUMMARY

Recently, attention has been drawn to the consideration of obtaining nonparametric Bayes estimates of a distribution function assuming a manageable prior (resulting in a manageable posterior distribution) on the space of distribution functions F on $R = (-\infty,\infty)$. Towards this goal, Ferguson [4] introduced a class of priors, known as Dirichlet process priors, on F which enjoy the property that the posterior distribution is again a Dirichlet process. Ferguson used this fact to obtain the Bayes estimator of the right sided cumulative distribution function F ($F(x)$ denotes the probability in (x,∞) and this useful convention is borrowed from Efron [3]) under a weighted squared error loss function. It can be readily seen that this Bayes estimator of F will have all the asymptotic properties enjoyed by the maximum likelihood estimator of F if there is no prior on F and the observations are i.i.d. with an unknown right c.d.f. F_0.

An important problem in survival analysis is that of estimating either parametrically or nonparametrically the survival curve $P(X > x) = F(x)$. (See, for example, Gross and Clark [6]). While treating this problem of estimating survival curves based on incomplete data, the authors [11] obtained the Bayes estimator of F under a weighted squared error loss function when the independent observations from F are randomly censored on the right under Dirichlet process priors of Ferguson [4]. They demonstrated that this Bayes estimator is an extension of the above mentioned Bayes estimator of Ferguson [4] and in a certain sense, also of the well-known Kaplan-Meier (KM) estimator [7] which maximizes the likelihood of the observations. Efron [3] and in a more detailed manner, Breslow and Crowley [1] showed that the KM estimator is weakly consistent and asymptotically normal

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under the assumption that all the censoring random variables are i.i.d. continuous random variables.

The object of this paper is to show that our Bayes estimator has good limiting properties including mean-square consistency (m.s.c.), almost sure consistency (a.s.c.) and asymptotic normality assuming that the observations are i.i.d. with right c.d.f. F_0 and that the censoring random variables are i.i.d. with a continuous distribution function. Efron [3] and Breslow and Crowley [1] have neither rate of convergence results for their weak consistency nor do they have m.s.c., while we obtain rates for both m.s.c. and a.s.c. Our methods of proof, in contrast with those of Breslow and Crowley [1], involve the analysis of the expectation and the variance of the logarithm of $W_n(u)$ involved in the Bayes estimator given in (2.2).

This paper, therefore, generates a new class of estimators of F_0, one for each parameter of the Dirichlet process, when the observations are censored on the right. Furthermore, we have shown that each member of the class has better asymptotic properties than those established for the KM estimator by Breslow and Crowley [1]. For results of this type in parametric cases, see the bibliographies of Lindley [8] and Shapiro [9]. A justification for this paper is given on page 615 of Ferguson [5]. In his words, "Bayes rules are certainly desirable since generally they are admissible and have nice large sample properties. Therefore, it behooves the statistician to suggest large classes of easily computable Bayes rules in the hope that users may find some rules to their liking," especially if the Bayes rules have the usual large sample properties enjoyed by the maximum likelihood estimator.

The next section formally describes the precise statement of the problem and needed assumptions. The rest of the sections deal with various asymptotic aspects of our Bayes estimator.

2. DESCRIPTION OF THE PROBLEM AND SOME NOTATION

Let $X_1,\ldots,X_n$ be a random sample from a right sided c.d.f. F with $F(0) = 1$ and $Y_1,\ldots,Y_n$ be another random sample such that $(X_1,\ldots,X_n)$ and $(Y_1,\ldots,Y_n)$ are mutually independent of each other. Set

$$\delta_i = [X_i \le Y_i] \quad \text{and} \quad Z_i = \min\{X_i, Y_i\} \tag{2.1}$$

for $i = 1,\ldots,n$ and assume that $1 - F$ be distributed according to Dirichlet process with parameter measure α on the Borel σ-field $\mathcal{B}$ in $(0,\infty)$. Then the Bayes estimator of F, under the loss function, $L(F,\hat{F}) = \int_0^\infty (F(u)-\hat{F}(u))^2 dw(u)$, is shown in [11] to be

$$\hat{F}_\alpha(u) = \frac{\alpha(u) + N^+(u)}{\alpha(R^+) + n} \prod_{i=1}^{n} \left\{ \frac{\alpha(Z_i^-) + N^+(Z_i) + \lambda_i}{\alpha(Z_i^-) + N^+(Z_i)} \right\}^{[\delta_i = 0,\, Z_i \le u]} \tag{2.2}$$

$$= B_n(u) \; W_n(u)$$

where $N^+(t) = \#$ of observations $> t$ and $\lambda_i = \#$ of observations at Z_i, $i = 1,\ldots,n$.

The main results of this paper concern the asymptotic behavior of $\hat{F}_\alpha$ under the following assumptions:

(A1) $X_1,\ldots,X_n$ are i.i.d. with right sided c.d.f. F_0, a fixed unknown distribution on $(0,\infty)$

(A2) $Y_1,\ldots,Y_n$ are i.i.d. with right c.d.f. G, a fixed unknown continuous distribution on $(0,\infty)$.

Thus, while the rule under consideration is a Bayes rule, the asymptotic properties of $\hat{F}_\alpha$ are obtained in a non-decision theoretic setup. In other words, we obtain the asymptotic behavior of $\hat{F}_\alpha$ as an estimator of F_0.

Throughout we assume that for a fixed (but otherwise arbitrary) u,

$$\text{(A3)} \qquad \alpha(u) > 0.$$

Since it is not possible to estimate $F_0(u)$ whenever $G(u) = 0$, we assume throughout that $G(u) > 0$ without further reference.

In the hope of reducing repetition, a right sided c.d.f. is referred to as simply a distribution and the assumptions stated above are assumed throughout the paper.

3. MEAN SQUARE CONSISTENCY WITH RATES

This property of $\hat{F}_\alpha$ can be studied through the corresponding one for the logarithm of $W_n(u)$ of (2.2). Under (A3), one obtains that

$$\text{(3.1)} \qquad \ln W_n(u) = \sum_{i=1}^{n} [\delta_i = 0,\ Z_i \leq u] \ \ln \left\{ \frac{\alpha(Z_i^-) + N^+(Z_i) + 1}{\alpha(Z_i^-) + N^+(Z_i)} \right\}$$

where $\alpha(s^-)$ = limit of $\alpha(t)$ as $t \uparrow s$. Observe that (3.1) (and hence (2.2)) is well-defined since $\alpha(u^-) > 0$ by (A3). This property is not enjoyed by the KM estimate which is not always well-defined in the right tail. It is precisely this property of converting (2.1) into a sum by use of logarithms that allows us to obtain stronger convergence results than do Breslow and Crowley [1] for the KM estimator.

For dealing with the expectation and the variance (and properties based on these) of $\ln W_n$, the following decomposition which follows by a logarithmic expansion of the summands in (3.1) and the succeeding lemmas will be extremely useful:

$$(3.2)\qquad \ln W_n(u) = R_{n,1}(u) + R_{n,2}(u) + R_{n,3}(u)$$

where

$$(3.3)\qquad nR_{n,1}(u) = \sum_{j=1}^{n}[\delta_j = 0,\ Z_j \le u]H^{-1}(Z_j),$$

$$(3.4)\qquad R_{n,2}(u) = \sum_{j=1}^{n}[\delta_j = 0,\ Z_j \le u]\sum_{\ell=2}^{\infty}\ell^{-1}(\alpha(Z_j^-)+1+N^+(Z_j))^{-\ell},$$

and

$$(3.5)\qquad nR_{n,3}(u) = \sum_{j=1}^{n}[\delta_j = 0,\ Z_j \le u]\{n[(\alpha(Z_j^-)+1+N^+(Z_j)]^{-1}-H^{-1}(Z_j)\}\ .$$

where

$$(3.6)\qquad H = F_0 G\ .$$

Lemma 3.1. $E[R_{n,1}(u)] = \ln G(u)$ and $nH^2(u)\,\mathrm{var}(R_{n,1}(u)) \le 1$.

Proof. Since the summands of $R_{n,1}(u)$ are identically distributed,

$$E[R_{n,1}(u)] = E[[\delta_1 = 0,\ Z_1 \le u]H^{-1}(Z_1)] = -\int_0^u F_0(t)H^{-1}(t)dG(t) = -\ln G(u)$$

by the definition of H in (3.6). The variance result follows since each of the iid summands in $R_{n,1}(u)$ is bounded by $H^{-1}(u)$.

Lemma 3.2. $\alpha^2(u^-)H^4(u)\binom{n+3}{4}E[R_{n,2}^2(u)] \le n^2(1+\alpha(u^-))^2$.

Proof. Since $(a_1+\ldots+a_n)^2 \le n\sum_{i=1}^{n} a_i^2$ for any real numbers $a_1,\ldots,a_n$, and since for fixed n, the summands of $R_{n,2}(u)$ are identically distributed,

$$(3.7)\qquad \begin{aligned} n^{-2}E[R_{n,2}^2(u)] &\le E[[\delta_1 = 0, Z_1 \le u](\sum_{\ell=2}^{\infty}\ell^{-1}(\alpha(Z_1^-)+1+N^+(Z_1))^{-\ell})^2] \\ &\le \{\sum_{\ell=2}^{\infty}(\alpha(u^-)+1)^{2-\ell}\}^2 E[(\alpha(u^-)+1+N^+(u))^{-4}] \end{aligned}$$

where the second inequality follows by bounding the series by

$\{\sum_{\ell=2}^{\infty}(\alpha(u^-)+1)^{2-\ell}\}\{\alpha(u^-)+1+N^+(u)\}^{-2}$ and by dropping the indicator function.

The result now follows from the following inequality and (3.7)

$$E[(\alpha(u^-)+1+N^+(u))^{-4}] = \sum_{k=0}^{n-1}\binom{n-1}{k}(\alpha(u^-)+1+k)^{-4}H^k(u)(1-H(u))^{n-1-k}$$

$$\leq H^{-4}(u)\sum_{k=0}^{n-1}\binom{n+3}{k+4}H^{k+4}(u)(1-H(u))^{n+3-(k+4)}/\binom{n+3}{4}$$

where the inequality follows since $k+i \leq i(\alpha(u^-)+k+1)$ for $i = 1,2,3$, and 4.

Lemma 3.3.

a) $nH^2(u)|E[R_{n,3}(u)]| \leq \alpha(R^+)+(1-H(u))^n(n+\alpha(R^+))+2\sqrt{n}\alpha(R^+)(1-H(u)^{1/2}(n+1)^{-1}$
$+ 2\alpha(R^+)(2+\alpha(R^+))\{n(n+1)\}^{-1}$

b) $(n+2)H^2(u)E[R^2_{n,3}(u)] \leq 2\{1-H(u)+2(n+2)^{-1}(\alpha^2(R^+)+(1-H(u))^2)\}$.

Proof. Since the summands of $R_{n,3}(u)$ are identically distributed,

$$E[R_{n,3}(u)] = E[[\delta_1=0,Z_1\leq u]\{n(\alpha(Z_1^-)+1+N^+(Z_1))-H^{-1}(Z_1)\}]$$

(3.8)
$$= -\int_0^u F_0(t)E[n(\alpha(t^-)+1+K_t)^{-1}-H^{-1}(t)]dG(t)$$

where K_t is a binomial random variable with parameters $n-1$ and $H(t)$.

Now observe that after some simplification,

$$H(t)E[n(\alpha(t^-)+1+K_t)^{-1}-H^{-1}(t)] = \sum_{k=0}^{n-1}\binom{n-1}{k}(k+1)^{-1}\{nH(t)-\alpha(t^-)-k-1\}$$

(3.9)
$$\times H^k(t)(1-H(t))^{n-k-1}+\sum_{k=0}^{n-1}\binom{n-1}{k}\alpha(t^-)\{k+1+\alpha(t^-)-nH(t)\}$$

$$\times H^k(t)(1-H(t))^{n-1-k}\{(k+1)(k+1+\alpha(t^-))\}^{-1} = I + II.$$

By a rearrangement,

$$-nH(t)I = \alpha(t^-)+(1-H(t))^n(nH(t)+\alpha(t^-))$$

while

$$II \leq \frac{2\alpha(t^-)}{n(n+1)} \sum_{k=0}^{n-1} \binom{n+1}{k+2} |k+2-(n+1)H(t)+H(t)+\alpha(t^-)-1| H^k(t)(1-H(t))^{n+1-(k+2)}$$

since $k+2 \leq 2(\alpha(t^-)+k+1)$. By a change of variable $(k+2 = \ell)$ and by using the binomial moments, we can show from the above inequality that

$$II \leq \frac{2\alpha(t^-)}{H\ (t)} \left\{ \frac{\{H(t)(1-H(t)\}^{1/2}}{\sqrt{n}(n+1)} + \frac{H(t)+\alpha(t^-)+1}{n(n+1)} \right\} .$$

This bound on II together with that on I, (3.8) and (3.9) give the first result since $t \leq u$ in all the calculations after (3.8).

Using the inequality $(\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} a_i^2$ and the fact that the summands of $R_{n,3}(u)$ are identically distributed for fixed n, it can be shown that $E[R_{n,3}^2(u)] \leq -\int_0^u F_0(t)E[(n(\alpha(t^-)+K_t+1)^{-1}-H^{-1}(t))^2]dG(t)$ where K_t is as in the proof of a). From here on, the proof runs parallel to that in a).

Two consequences of the decomposition (3.2) and the above three lemmas are given below, the first of which conserns the mean square consistency (m.s.c.) of $\ln W_n$ as an estimator of $\ln G^{-1}$ while the second one concerns the m.s.c. of $\hat{F}_\alpha$ of (2.2) as an estimator of F_0.

Theorem 3.1. Let $F_0(u) > 0$. $nE[|\ln W_n(u)-\ln G^{-1}(u)|^2]$ is bounded.

Proof. The proof is a direct consequence of the decomposition (3.2) and the above three lemmas.

Theorem 3.2. Let $F_0(u) > 0$. Then $nE[(\hat{F}_\alpha(u)-F_0(u))^2]$ is bounded, where the bound is given in the remark below.

Proof. Recalling that B_n and W_n are defined in (2.2), we obtain by a C_r-inequality,

$$(3.10)\qquad 2^{-1}(\hat{F}_\alpha(u)-F_0(u))^2 \le G^{-2}(u)(B_n(u)-H(u))^2+B_n^2(u)(W_n(u)-G^{-1}(u))^2$$

where we used the equality $H = F_0G$. Since $nB_n(u)$ can be approximated by the binomial random variable $N^+(u)$ whose expectation is $nH(u)(=nF_0(u)G(u))$, we obtain that

$$(3.11)\qquad (\alpha(R^+)+n)^2E[(B_n(u)-H(u))^2] = (\alpha(u)-H(u)\alpha(R^+))^2+nH(u)(1-H(u)) .$$

After writing $e^{\ln W_n(u)}$ and $e^{-\ln G(u)}$ for $W_n(u)$ and $G^{-1}(u)$, respectively, and then using the mean value theorem leads to $(W_n(u)-G^{-1}(u))^2 \le |\ln W_n(u)-\ln G^{-1}(u)|^2(W_n(u)+G^{-1}(u))^2$ since $W_n(u)$ and $G^{-1}(u) \ge 1$. Hence,

$$(3.12)\qquad E[B_n^2(u)(W_n(u)-G^{-1}(u))^2] \le (1+G^{-1}(u))^2E[|\ln W_n(u)-\ln G^{-1}(u)|^2]$$

since B_n and B_nW_n are ≤ 1. Using the fact that $E[|A+B+C|^2] \le 3(E[A^2] + E[B^2] + E[C^2])$ and then using Lemmas 3.1, 3.2, and 3.3 give the result in view of (3.10) - (3.12).

Remark. The bound in Theorem 3.2 is given by

$$\begin{aligned} C_n(u) = 2\{&\alpha(R^+)+n\}^{-2}\{n^2H(u)(1-H(u))+n(\alpha(u)-\alpha(R^+)H(u))^2\} \\ &+ 6(1+G^{-1}(u))^2H^{-2}(u)\{1+\alpha^{-2}(u^-)H^{-2}(u)\tbinom{n+3}{4}^{-1}n(1+\alpha(u^-))^2 \\ &+ \text{the rhs of b) in Lemma 3.3 + the square of the rhs of} \\ &\quad \text{a) in Lemma 3.3}\}. \end{aligned}$$

4. ALMOST SURE CONSISTENCY

Looking at the estimator $\hat{F}_\alpha$ of (2.2), it is obvious that $\hat{F}_\alpha$ converges a.s. provided $W_n(u)$ does likewise. The following lemma concerning the almost sure behavior of $R_{n,1}$, $R_{n,2}$, and $R_{n,3}$ involved in $\ln W_n(u)$ of (3.2) is essential for the main result of this section.

Lemma 4.1. Let $F_0(u) > 0$. Then

a) $$|R_{n,1}(u) + \ln G(u)| = 0(\frac{\log \log n}{\sqrt{n}}) \text{ a.s.}$$

b) $$|R_{n,2}(u)| = 0(\frac{\log n}{\sqrt{n}}) \text{ a.s.}$$

c) $$|R_{n,3}(u)| = 0(\frac{\log n}{\sqrt{n}}) \text{ a.s.}$$

Proof. a) follows from the first part of Lemma 3.1 and the law of iterated logarithm for iid random variables. b) follows from Lemma 3.2 and the Glivenko-Cantelli theorem since $\sum_{n=2}^{\infty} n^{-1}(\log n)^{-2} < \infty$.

To prove c), we observe that $|R_{n,3}(u)|$ is exceeded by

$$\{n^{-1} \sum_{j=1}^{n} [\delta_j = 0,\ Z_j \leq u]\} \sup_{0 < t \leq u} \{|\frac{n}{\alpha(t^-)+1+N^+(t)} - \frac{1}{H(t)}|\} .$$

Since the expression in the first curly brackets is bounded by unity, it is enough to show that

$$\text{(4.1)} \qquad \sup_{0 < t \leq u} \{|\frac{n}{\alpha(t^-)+1+N^+(t)} - \frac{1}{H(t)}|\} = 0(\frac{\log n}{\sqrt{n}}) \text{ a.s.}$$

The lhs of (4.1) is bounded by

$$\text{(4.2)} \qquad \frac{1}{H(u)} \frac{n}{\alpha(u^-)+1+N^+(u)} \sup_{0 < t \leq u} \{|\frac{N^+(t)}{n} - H(t) - \frac{1+\alpha(t^-)}{n}|\} .$$

Now by Lemma 1 of Dvoretzky, Kiefer, and Wolfowitz [2]

$$P[\frac{\sqrt{n}}{\log n} \sup_{0 < t \leq u} \{|\frac{N^+(t)}{n} - H(t)|\} > \varepsilon] \leq c_0\, e^{-c(\log n)^2}$$

for some absolute constants c_0 and c. Hence

$$\sup_{0 < t \le u} \{|\frac{N^+(t)}{n} - H(t)|\} = O(\frac{\log n}{\sqrt{n}}) \quad \text{a.s.}$$

since $c(\log n)^2 \geq 2 \log n$ for large n and $\sum_{n=1}^{\infty} n^{-2} < \infty$. This completes the proof of (4.1) and hence also of c).

Theorem 4.1. Let $F_0(u) > 0$. Then

$$\hat{F}_\alpha(u) - F_0(u) = O(\log n/\sqrt{n}) \quad \text{a.s.}$$

Proof. By a triangle inequality,

$$(4.3) \qquad |\hat{F}_\alpha(u)-F_0(u)| \le G^{-1}(u)|B_n(u)-H(u)|+B_n(u)|W_n(u)-G^{-1}(u)|.$$

The first term on the rhs is $O(\log\log n/\sqrt{n})$ a.s. since $|(N^+(u)/n)-H(u)| = O(\log\log n/\sqrt{n})$ a.s. by the law of iterated logarithm.

As in the proof of Theorem 3.2, we can show that

$$(4.4) \qquad B_n(u)|W_n(u)-G^{-1}(u)| \le (1+G^{-1}(u))|\ln W_n(u)-\ln G^{-1}(u)|.$$

But $|\ln W_n(u)-\ln G^{-1}(u)| = O(\log n/\sqrt{n})$ a.s. due to the decomposition (3.2), a triangle inequality, and parts a), b), and c) of Lemma 4.1. Thus (4.3) and (4.4) complete the proof.

5. WEAK CONVERGENCE OF $\hat{F}_\alpha$

In this section, we consider the weak convergence of $\{\hat{F}_\alpha(u)|0 < u \le T\}$ where $T < \infty$. We assume throughout this section that $H(T) = F_0(T)G(T) > 0$. It is convenient sometimes to suppress the dependence of the functions and to let $\| \|_T$ denote the sup norm over $(0,T]$.

The discussion to follow reduces the consideration of $\{\hat{F}_\alpha(u)|0 < u \le T\}$ to a much more accessible form. To start with, we observe that

$$(5.1) \qquad \sqrt{n}(\hat{F}_\alpha - F_0) = \sqrt{n}(B_n - H)G^{-1} + \sqrt{n}(W_n - G^{-1})B_n$$

and that

$$W_n - G^{-1} = e^{\ln W_n} - e^{\ln G^{-1}} = e^{\ln G^{-1}}(e^{\ln W_n - \ln G^{-1}} - 1)$$

$$= G^{-1}(\ln W_n - \ln G^{-1}) + \frac{(\ln W_n - \ln G^{-1})^2}{2} e^{c + \ln G^{-1}}$$

where c is between 0 and $\ln W_n - \ln G^{-1}$. Therefore, from (5.1), we

$$\| \sqrt{n}(\hat{F}_\alpha - F_0) - \sqrt{n}G^{-1}(B_n - H) - \sqrt{n}\, HG^{-1}(\ln W_n - \ln G^{-1} \|_T$$

$$(5.2) \qquad \leq \sqrt{n}\, \|B_n - H\|_T \, \|\ln W_n - \ln G^{-1}\|_T \, \|G^{-1}\|_T + \frac{\sqrt{n}}{2}(\|\ln W_n - \ln G^{-1}\|_T)^2.$$

The purpose of the following lemma is to show that the rhs of (5.2) $\to 0$ a.s. by showing that $n^\beta \|\ln W_n - \ln G^{-1}\|_T \to 0$ a.s. for any $2\beta < 1$.

<u>Lemma 5.1.</u> $n^\beta \|\ln W_n - \ln G^{-1}\|_T \to 0$ a.s. for any $2\beta < 1$.

<u>Proof</u>. With $\tilde{H}_n(u) = \int_0^u F_0(t)d(1-G(t)) = P[\delta_1 = 0,\ Z_1 \leq u]$ and with $n\tilde{H}_n(u) = \sum_{j=1}^n [\delta_j = 0,\ Z_j \leq u]$, we have

$$\|\ln W_n - \ln G^{-1}\|_T = \| \int_0^{\cdot} n \ln\{1 - (\alpha(s^-) + 1 + H_n(s))^{-1}\} d\tilde{H}_n(s) - \int_0^{\cdot} H^{-1}(s) d\tilde{H}(s) \|_T$$

$$(5.3) \qquad \leq \| \int_0^{\cdot} n\{\alpha(s^-) + 1 + nH_n(s)\}^{-1} d\tilde{H}_n(s) - \int_0^{\cdot} H^{-1}(s) d\tilde{H}(s) \|_T$$

$$+ \frac{n}{(\alpha(T^-) + 1 + nH_n(T))^2} \frac{(1+\alpha(T^-))^2}{(\alpha(T^-))^2}$$

where the inequality follows by a logarithmic expansion and an obvious weakening of the series from the second term onwards. Observe that the second term in the rhs of (5.3) $\to 0$ a.s. at a rate $n^{-\beta}$ with $2\beta < 1$.

For the first term on the rhs of (5.3),

$$\| \int_0^{\cdot} n\{\alpha(s^-)+1+nH_n(s)\}^{-1} d\tilde{H}_n(s) - \int_0^{\cdot} H^{-1}(s)d\tilde{H}(s) \|_T$$

$$(5.4) \qquad \leq \| \int_0^{\cdot} \{n\{\alpha(s^-)+1+nH_n(s)\}^{-1}-H^{-1}(s)\}d\tilde{H}_n(s) \|_T + \| \int_0^{\cdot} H^{-1}(s)d(\tilde{H}_n-\tilde{H})(s) \|_T$$

$$\leq \| \int_0^{\cdot} [n\{\alpha(s^-)+1+nH_n(s)\}^{-1}-H^{-1}(s)]d\tilde{H}_n(s) \|_T + 2H^{-1}(T) \| \tilde{H}_n-\tilde{H} \|_T$$

where the second inequality follows by applying integration by parts to $\int_0^{\cdot} H^{-1}(s)d(\tilde{H}_n-\tilde{H})(s)$ and upon observing that the variation of H^{-1} on $(0,\cdot]$ is $H^{-1}(\cdot)$. By the law of iterated logarithm for iid random variables $\|n\{\alpha(\cdot^-)+1+nH_n(\cdot)\}-H^{-1}(\cdot)\|_T \to 0$ a.s. at a rate $O(n^{-\beta})$ with $2\beta < 1$ and by applying the argument given by Singh [10] to the random variables $[\delta_j = 0, Z_j \leq u]$, $j = 1,\ldots,n$, we obtain that $\|\tilde{H}_n-\tilde{H}\|_T \to 0$ a.s. at a rate $O(n^{-\beta})$ with $2\beta < 1$. Consequently the rhs of (5.4) and hence, the rhs of (5.3) $\to 0$ a.s. at a rate $O(n^{-\beta})$ with $2\beta < 1$.

In view of (5.2), an easy Corollary to the above lemma is

Corollary 5.1. $\|\sqrt{n}(\hat{F}_\alpha-F_0)-\sqrt{n}G^{-1}(B_n-H)-\sqrt{n}HG^{-1}(\ln W_n-\ln G^{-1})\|_T \to 0$ a.s. at a rate $O(n^{-\beta})$ for any $2\beta < 1$.

By following the method of proof of Lemma 5.1, we can also show that

Lemma 5.2. $\|\sqrt{n}(\ln W_n-\ln G^{-1})-\sqrt{n}(\int_0^{\cdot} H_n^{-1}d\tilde{H}_n - \int_0^{\cdot} H^{-1}d\tilde{H})\|_T \to 0$ a.s. at a rate $O(n^{-\beta})$ with $2\beta < 1$.

Note. The random integral $\int_0^{\cdot} H_n^{-1} d\tilde{H}_n$ could be infinity, but finite a.s. since $P[H_n(s) = 0] = P[\delta_j = 0, Z_j \le s \text{ for } j = 1,\ldots,n] \le P[Z_j \le T \text{ for } j = 1,\ldots,n] = (1-H(T))^n$ for all $s \le T$.

Hence, by Corollary 5.1 and Lemma 5.2, we can study the weak convergence of $\{\hat{F}_\alpha(u) | 0 < u \le T\}$ through the corresponding one for

$$\sqrt{n}G^{-1}(B_n - H) + \sqrt{n}HG^{-1}\left(\int_0^{\cdot} H_n^{-1} d\tilde{H}_n - \int_0^{\cdot} H^{-1} dH\right) . \tag{5.5}$$

The following theorem, which is parallel to Theorem 3 of Breslow and Crowley [1], is needed in the study of the weak convergence of (5.5) or equivalently that of $\{\hat{F}_\alpha(u) | 0 < u \le T\}$.

Theorem 5.1. Define $(P_n, Q_n) \in D(0,T) \times D(0,T]$ ($D(0,T]$ is the space of functions on $(0,T]$ with jump discontinuities) by $P_n = \sqrt{n}(H - H_n)$ and $Q_n = \sqrt{n}(\tilde{H}_n - \tilde{H})$. Then (P_n, Q_n) converges weakly to a bivariate Gaussian process (P,Q) which has mean 0 and a covariance structure given for $s \le t$ by

$$\left\{\begin{array}{rcl} \mathrm{Cov}(P(s),P(t)) & = & H(t)(1-H(s)) \quad , \\ \mathrm{Cov}(Q(s),Q(t)) & = & \tilde{H}(s)(1-\tilde{H}(t)) \quad , \\ \mathrm{Cov}(P(s),Q(t)) & = & \tilde{H}(s)-\tilde{H}(t)(1-H(s)), \\ \text{and} \quad \mathrm{Cov}(Q(s),P(t)) & = & \tilde{H}(s)H(t) \end{array}\right. \tag{5.6}$$

where $H = F_0 G$ and $\tilde{H} = \int_0^{\cdot} F_0 d(1-G)$.

As in (7.9) of Breslow and Crowley [1], we can represent (5.5) as

$$(5.5) = \sqrt{n}G^{-1}(B_n - H) + A_n + B_n + R_{1,n} + R_{2,n} \tag{5.7}$$

where

$$(5.8)\quad \begin{cases} A_n = \int_0^{\cdot} P_n H^{-2} d\tilde{H} , \\ B_n = Q_n H^{-1} - \int_0^{\cdot} Q_n H^{-2} d(1-H) , \\ R_{1,n} = n^{-1/2} \int_0^{\cdot} P_n^2 H^{-2} H_n^{-1} d\tilde{H}_n , \\ \text{and} \quad R_{2,n} = \int_0^{\cdot} P_n H^{-1} H_n^{-1} d(\tilde{H}_n - \tilde{H})(u) . \end{cases}$$

By the above representation for (5.5) and steps similar to Theorem 4 of Breslow and Crowley [1] , we obtain the following theorem which is similar to Theorem 5 of Breslow and Crowley [1].

Theorem 5.2. Let $T < \infty$ and $H(T) > 0$. Let F_0 and G be continuous. Then the random function $\sqrt{n}(\hat{F}_\alpha - F_0)$ on $(0,T]$ converges weakly to a mean 0 Gaussian process $R^* = -G^{-1}P + \int_0^{\cdot} H^{-2}Pd\tilde{H} + H^{-1}Q + \int_0^{\cdot} H^{-2}QdH$ with covariance structure given for $s \leq t$ by

$$(5.9)\quad \begin{aligned} \mathrm{Cov}(R^*(s),R^*(t)) &= F_0(s)F_0(t)\{H^{-1}(s)(1-H(s)) + \int_0^s H^{-1}G^{-1}dG\} \\ &= F_0(s)F_0(t)\{\int_0^s H^{-1}F_0^{-1}d(1-F_0)\} . \end{aligned}$$

Remark 5.1. The covariance calculations involved in (5.9) are given in the Appendix. We notice here that the rhs of (5.9) coincides with (7.13) of Breslow and Crowley [1].

CONCLUDING REMARKS

There are three small sample advantages for the Bayes estimator (2.2) over the KM estimator. The first is that it is defined everywhere on the real line for any n while KM estimator is not. Secondly, as illustrated in our paper [11], the Bayes estimator is smoother than the KM estimator.

The final important advantage is that the Bayes estimator is an admissible estimator of F provided that the support of α = support of $w = (0,\infty)$ under the loss function $L(F,\hat{F}) = \int_0^\infty (F(u)-\hat{F}(u))^2 dw(u)$ and under the weak convergence topology.

The results of this paper can be extended to the case in which $Y_1,\ldots,Y_n$ are independent, but not identically distributed. The technique used to obtain this extension is different from the ones proposed here and will appear elsewhere.

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APPENDIX: COVARIANCE STRUCTURE OF R^* OF THEOREM 5.2

To study the covariance structure of R^*, it is convenient to study the covariance structure of

$$R = F_0^{-1}R^* = H^{-1}P + \int_0^{\cdot} H^{-2}Pd\tilde{H} + H^{-1}Q + \int_0^{\cdot} H^{-2}QdH.$$

We use repeatedly (5.6), integration by parts and the equalities $H = F_0G$ and $d\tilde{H} = d(\int_0^{\cdot} F_0 d(1-G)) = -F_0 dG$. We write Cov(R(s),R(t) = Var(R(s)) + Cov(R(s),R(t)-R(s)) for $0 < s \leq t < T$, show that Var(R(s)) = the expression in the curly brackets of (5.9), and that Cov(R(s),R(t)-R(s)) = 0.

VARIANCE OF R(s)

(1) $$\mathrm{Var}(H^{-1}(s)P(s)) = H^{-1}(s)(1-H(s)) ,$$

(2) $$2\mathrm{Cov}(H^{-1}(s)P(s),H^{-1}(s)Q(s)) = -2H^{-1}(s)\tilde{H}(s) ,$$

(3) $$2\mathrm{Cov}(H^{-1}(s)P(s), \int_0^s H^{-2}Pd\tilde{H}) = 2H^{-1}(s) \int_0^s \frac{\mathrm{Cov}(P(u),P(s))}{H^2(u)} d\tilde{H}$$
$$= -2 \int_0^s \frac{d\tilde{H}}{H^2} - 2 \ln G(s) ,$$

(4) $$2\mathrm{Cov}(H^{-1}(s)P(s), \int_0^s QH^{-2}dH) = \frac{2}{H(s)} \int_0^s \frac{\mathrm{Cov}(P(s),Q(u))}{H^2(u)} dH$$
$$= + 2 \frac{\tilde{H}(s)}{H(s)} + 2 \ln G(s) ,$$

(5) $$\mathrm{Var}(\int_0^s H^{-2}Pd\tilde{H}) = 2 \int_0^s \int_0^u \frac{\mathrm{Cov}(P(u),P(r))}{H^2(u)H^2(r)} d\tilde{H}(r)d\tilde{H}(u)$$
$$= 2 \int_0^s \frac{1-H(r)}{H(r)} \{\int_r^s \frac{d\tilde{H}(u)}{H(u)}\} d\tilde{H}(r) = 2 \int_0^s \frac{(\ln G(s)-\ln G(r))}{HG} dG - \ln^2 G(s),$$

(6) $$\mathrm{Var}(H^{-1}(s)Q(s)) = \frac{\tilde{H}(s)(1-\tilde{H}(s))}{H^2(s)}$$

(7) $$\mathrm{Var}\Big(\int_0^s H^{-2}Q\,dH\Big) = 2\int_0^s\int_0^u \frac{\mathrm{Cov}(Q(u),Q(r))}{H^2(u)H^2(r)}\,dH(u)dH(r)$$

$$= 2\int_0^s\int_0^u \frac{\tilde{H}(r)(1-\tilde{H}(u))}{H^2(u)\,H^2(r)}\,dH(r)dH(u) = 2\int_0^s \frac{1-\tilde{H}}{H^2}\Big\{-\frac{\tilde{H}}{H} - \ln G\Big\}dH$$

$$= \frac{(1-\tilde{H}(s))\tilde{H}(s)}{H^2(s)} - \int_0^s \frac{d\tilde{H}-2\tilde{H}d\tilde{H}}{H^2} + 2\left\{\frac{(1-\tilde{H}(s))}{H(s)}\ln G(s) - \int_0^s \frac{1-\tilde{H}}{HG}\,dG + \int_0^s \frac{\ln G}{H}\,d\tilde{H}\right\}$$

(8) $$2\mathrm{Cov}\Big(\int_0^s H^{-2}P\,d\tilde{H}, H^{-1}(s)Q(s)\Big) = \frac{2}{H(s)}\int_0^s \frac{\mathrm{Cov}(Q(s),P(u))}{H^2(u)}\,d\tilde{H}(u)$$

$$= \frac{2}{H(s)}\left\{\int_0^s \frac{\tilde{H}}{H^2}\,d\tilde{H} - \tilde{H}(s)\int_0^s \frac{1}{H^2}\,d\tilde{H} + \tilde{H}(s)\int_0^s \frac{1}{H}\,d\tilde{H}\right\}$$

(9) $$2\mathrm{Cov}\Big(H^{-1}(s)Q(s), \int_0^s H^{-2}Q\,dH\Big) = \frac{2}{H(s)}\int_0^s \frac{\mathrm{Cov}(Q(u),Q(s))}{H^2(u)}\,dH(u)$$

$$= -\frac{2(1-\tilde{H}(s))\tilde{H}(s)}{H^2(s)} - \frac{2(1-\tilde{H}(s))}{\tilde{H}(s)}\ln G(s)$$

(10) $$2\mathrm{Cov}\Big(\int_0^s H^{-2}P\,d\tilde{H}, \int_0^s H^{-2}Q\,dH\Big)$$

$$= 2\int_0^s\int_0^u \frac{\mathrm{Cov}(P(u),Q(r))}{H^2(u)H^2(r)}\,dH(r)d\tilde{H}(u) + 2\int_0^s\int_0^u \frac{\mathrm{Cov}(P(r),Q(u))}{H^2(u)H^2(r)}\,d\tilde{H}(r)dH(u)$$

$$= 2\int_0^s\int_0^u \frac{\tilde{H}(r)H(u)}{H^2(u)H^2(r)}\,d\tilde{H}(u)dH(r) + 2\int_0^s\int_0^u \frac{\tilde{H}(r)-\tilde{H}(u)(1-H(r))}{H^2(u)H^2(r)}\,dH(r)d\tilde{H}(u)$$

$$= \frac{2}{H(s)} \int_0^s \frac{\tilde{H}}{HG}\, dG + 2\int_0^s \frac{\tilde{H}\, d\tilde{H}}{H^3} - \frac{2\tilde{H}(s)}{H(s)} \int_0^s \frac{dG}{HG} + \frac{2\tilde{H}(s)}{H(s)} \ln G(s) - 2\int_0^s \frac{\tilde{H}}{H^3}\, d\tilde{H}$$

$$+ 2\int_0^s \frac{\tilde{H}}{H^2}\, d\tilde{H} - 2\int_0^s \frac{(\ln G(s) - \ln G(r))}{HG}\, d\tilde{H} + 2\int_0^s \frac{\ln G(s) - \ln G(r)}{D}\, dG$$

$$-2\int_0^s \frac{\tilde{H}}{H^2}\, d\tilde{H} + \ln^2 G(s).$$

Adding (1) through (10) and using the facts that $H = F_0 G$ and $d\tilde{H} = -F_0 dG$ wherever necessary, we obtain the expression in the curly brackets of (5.9).

COVARIANCE OF $R(s)$ AND $R(t)-R(s)$

There are 16 terms in this covariance calculation which are grouped below into 5 sets of expressions. The sum of all the expressions in each group will be equal to zero, thus showing that $Cov(R(s), R(t)-R(s)) = 0$ for $0 < s \leq t < T$.

(A1) $$Cov\left(\frac{P(s)}{H(s)}, \frac{P(t)}{H(t)} - \frac{P(s)}{H(s)}\right) = \frac{Cov(P(t),P(s))}{H(t)H(s)} - \frac{Var(P(s))}{H^2(s)} = 0$$

(A2) $$Cov\left(\frac{P(s)}{H(s)}, \int_s^t H^{-2} P\, d\tilde{H}\right) = H^{-1}(s) \int_s^t \frac{Cov(P(u),P(s))}{H^2(u)}\, d\tilde{H} = \frac{1-H(s)}{H(s)}(\ln G(s) - \ln G(t))$$

(A3) $$Cov\left(\frac{P(s)}{H(s)}, \frac{Q(t)}{H(t)} - \frac{Q(s)}{H(s)}\right) = \frac{\tilde{H}(s)}{H(s)}\left\{\frac{1}{H(t)} - 1\right\} - \frac{\tilde{H}(t)(1-H(s))}{H(s)H(t)}$$

(A4) $$Cov\left(\frac{P(s)}{H(s)}, \int_s^t H^{-2} Q\, dH\right) = \frac{1}{H(s)} \int_s^t \frac{Cov(P(s),P(u))}{H^2(u)}\, dH = H^{-1}(s) \int_s^t \frac{\tilde{H}(s) - \tilde{H}(u)(1-H(s))}{H^2(u)}\, dH$$

$$= \frac{\tilde{H}(s)}{H(s)}\left\{\frac{1}{H(s)} - \frac{1}{H(t)}\right\} - \frac{(1-H(s))}{H(s)}\left\{\frac{\tilde{H}(s)}{H(s)} - \frac{\tilde{H}(t)}{H(t)} + \ln G(s) - \ln G(t)\right\}$$

(B1) $\operatorname{Cov}\left(\int_0^s H^{-2}Pd\tilde{H},\ \frac{P(t)}{H(t)} - \frac{P(s)}{H(s)}\right)$

$$= \frac{1}{H(t)}\int_0^s \frac{\operatorname{Cov}(P(u),P(t))}{H^2(u)}\,d\tilde{H}(u) - \frac{1}{H(s)}\int_0^s \frac{\operatorname{Cov}(P(u),P(s))}{H^2(u)}\,d\tilde{H}(u)$$

$$= \int_0^s \frac{1-H(u)}{H^2(u)}\,d\tilde{H}(u) - \int_0^s \frac{1-H(u)}{H^2(u)}\,d\tilde{H}(u) = 0 .$$

(B2) $\operatorname{Cov}\left(\frac{Q(s)}{H(s)},\ \frac{P(t)}{H(t)} - \frac{P(s)}{H(s)}\right) = \frac{\operatorname{Cov}(Q(s),P(t))}{H(s)H(t)} - \frac{\operatorname{Cov}(Q(s),P(s))}{H^2(s)} = 0.$

(B3) $\operatorname{Cov}\left(\int_0^s H^{-2}QdH,\ \frac{P(t)}{H(t)} - \frac{P(s)}{H(s)}\right) = \frac{1}{H(t)}\int_0^s \frac{\operatorname{Cov}(P(u),Q(t))}{H^2(u)}\,dH(u)$

$$- \frac{1}{H(s)}\int_0^s \frac{\operatorname{Cov}(Q(u),P(s))}{H^2(u)}\,dH(u) = 0.$$

(C1) $\operatorname{Cov}\left(\int_0^s H^{-2}Pd\tilde{H}, \int_s^t H^{-2}Pd\tilde{H}\right) = \int_0^s\int_0^t \frac{\operatorname{Cov}(P(u),P(r))}{H^2(u)H^2(r)}\,d\tilde{H}(u)d\tilde{H}(r)$

$$= \int_0^s \frac{(1-H(r))}{H^2(r)}\,d\tilde{H}(r) \int_s^t \frac{1}{H(u)}\,d\tilde{H}(u) = \int_0^s \frac{1-H(r)}{H^2(r)}\,(\ln G(s) - \ln G(t))\,d\tilde{H}(r)$$

$$= (\ln G(s) - \ln G(t)) \int_0^s \frac{1-H(r)}{H^2(r)}\,d\tilde{H}(r) .$$

(C2) $\operatorname{Cov}\left(\int_0^s H^{-2}PdH,\ \frac{Q(t)}{P(t)} - \frac{Q(s)}{Q(s)}\right) = \frac{1}{H(s)}\int_0^s \frac{\operatorname{Cov}(Q(t),P(u))}{H^2(u)} - \frac{1}{H(s)}\int_0^s \frac{\operatorname{Cov}(Q(s),P(u))}{H^2(u)}\,d\tilde{H}(u)$

$$= \frac{1}{H(t)}\int_0^s \frac{\tilde{H}(u)-\tilde{H}(t)(1-H(u))}{H^2(u)}\,d\tilde{H}(u) - \frac{1}{H(s)}\int_0^s \frac{\tilde{H}(u)-\tilde{H}(s)(1-H(u))}{H^2(u)}\,d\tilde{H}(u)$$

$$\frac{1}{H(t)}\int_0^s \frac{\tilde{H}}{H^2}\,d\tilde{H} - \frac{\tilde{H}(t)}{H(t)}\int_0^s \frac{1-H}{H^2}\,d\tilde{H} - \frac{1}{H(s)}\int_0^s \frac{\tilde{H}}{H^2}\,d\tilde{H} + \frac{\tilde{H}}{H}\int_0^s \frac{1-H}{H^2}\,d\tilde{H}$$

(C3) $$\operatorname{Cov}\left(\int_0^s H^{-2} P\,d\tilde{H},\ \int_s^t H^{-2} Q\,dH\right) = \int_0^s \int_s^t \frac{\operatorname{Cov}(Q(r),P(u))}{H^2(r)H^2(u)}\,dH(r)\,d\tilde{H}(u)$$

$$= \int_0^s \int_s^t \frac{\tilde{H}(u)-\tilde{H}(r)(1-H(u))}{H^2(u)H^2(r)}\,dH(r)\,d\tilde{H}(u)$$

$$= \left\{\frac{1}{H(s)} - \frac{1}{H(t)}\right\}\int_0^s \frac{\tilde{H}\,d\tilde{H}}{H^2} + \left\{\frac{\tilde{H}(t)}{H(t)} - \frac{\tilde{H}(s)}{H(s)} + \ln G(t) - \ln G(s)\right\}\int_0^s \frac{1-H}{H^2}\,d\tilde{H}$$

(D1) $$\operatorname{Cov}\left(H^{-1}(s)Q(s),\ \int_s^t H^{-2} P\,d\tilde{H}\right) = H^{-1}(s)\int_s^t \frac{\operatorname{Cov}(Q(s),P(u))}{H^2(u)}\,d\tilde{H}(u)$$

$$= \frac{\tilde{H}(s)}{H(s)}\left(\ln G(s) - \ln G(t)\right) .$$

(D2) $$\operatorname{Cov}\left(H^{-1}(s)Q(s),\frac{Q(t)}{H(t)} - \frac{Q(s)}{H(s)}\right) = \frac{\operatorname{Cov}(Q(s),Q(t))}{H(s)H(t)} - \frac{\operatorname{Cov}(Q(s),Q(s))}{H^2(s)}$$

$$= \frac{\tilde{H}(s)(1-\tilde{H}(t))}{H(s)H(t)} - \frac{\tilde{H}(s)(1-\tilde{H}(s))}{H^2(s)} .$$

(D3) $$\operatorname{Cov}\left(\frac{Q(s)}{H(s)},\ \int_s^t H^{-2} Q\,dH\right) = \frac{1}{H(s)}\int_0^t \frac{\operatorname{Cov}(Q(s),Q(u))}{H^2(u)}\,dH(u) = \frac{\tilde{H}(s)}{H(s)}\int_s^t \frac{1-\tilde{H}}{H^2}\,dH$$

$$= \frac{\tilde{H}(s)}{H(s)}\left\{\frac{1}{H(s)} - \frac{1}{H(t)} + \frac{\tilde{H}(t)}{H(t)} - \frac{\tilde{H}(s)}{H(s)} + \ln G(t) - \ln G(s)\right\}.$$

(E1) $$\operatorname{Cov}\left(\int_0^s H^{-2} Q\,dH,\ \int_s^t H^{-2} P\,d\tilde{H}\right) = \int_0^s \int_s^t \frac{\operatorname{Cov}(P(r),Q(u))}{H^2(r)H^2(u)}\,d\tilde{H}(r)\,dH(u)$$

$$= \int_0^s \int_s^t \frac{H(r)\tilde{H}(u)}{H^2(r)H^2(u)}\,d\tilde{H}(r)\,dH(u)$$

$$= -\{-\ln G(t) + \ln G(s)\}\left\{\frac{\tilde{H}(s)}{H(s)} + \ln G(s)\right\}.$$

(E2) $$\mathrm{Cov}\left(\int_0^s H^{-2}Q\,dH,\ \frac{Q(t)}{H(t)} - \frac{Q(s)}{H(s)}\right)$$

$$= \frac{1}{H(t)}\int_0^s \frac{\mathrm{Cov}(Q(t),Q(u))}{H^2(u)}\,dH(u) - \frac{1}{H(s)}\int_0^s \frac{\mathrm{Cov}(Q(s),Q(u))}{H^2(u)}\,dH(u)$$

$$= \frac{1-\tilde{H}(t)}{H(t)}\int_0^s \frac{\tilde{H}dH}{H^2} - \frac{1-\tilde{H}(s)}{H}\int_0^s \frac{\tilde{H}}{H^2}\,dH\ .$$

(E3) $$\mathrm{Cov}\left(\int_0^s H^{-2}Q\,dH, \int_s^t H^{-2}Q\,dH\right) = \int_0^s\int_s^t \frac{\mathrm{Cov}(Q(r),Q(u))}{H^2(u)H^2(r)}\,dH(u)dH(r)$$

$$= \int_0^s\left(\int_s^t \frac{1-\tilde{H}(u)}{H^2(u)}\,dH(u)\right)\frac{\tilde{H}(r)}{H(r)}\,dH(r)$$

$$= \left\{\frac{1}{H(s)} - \frac{1}{H(t)}\right\}\int_0^s \frac{\tilde{H}}{H^2}\,dH + \left(\int_0^s \frac{\tilde{H}}{H^2}\,dH\right)\left\{\frac{\tilde{H}(t)}{H(t)} - \frac{\tilde{H}(s)}{H(s)} + \ln G(t) - \ln G(s)\right\}.$$

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Let $X_1,\ldots,X_n$ be iid F and let $Y_1,\ldots,Y_n$ be independent (and independent also of $X_1,\ldots,X_n$) random variables. For i = 1,...,n, set $\delta_i = 1$ if $X_i \le Y_i$ and $= 0$ if $X_i > Y_i$, and $Z_i = \min\{X_i, Y_i\}$. Then assuming that F is distributed according to a Dirichlet process (Ferguson,

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(1973) Ann. Stat. 2, 209-230) with parameter α, the authors (1976), J. Amer. Stat. Assoc., 61, December) obtained the Bayes estimator $\hat{F}_\alpha$ of F under the loss function $L(F,\hat{F}) = \int(F(u)-\hat{F}(u))^2 dw(u)$ using $(\delta_1,Z_1),\ldots,(\delta_n,Z_n)$. Now let $X_1,\ldots,X_n$ be iid F_0 and $Y_1,\ldots,Y_n$ be iid G where both F_0 and G are unknown continuous distributions. In this paper, it is shown that $\hat{F}_\alpha$ is mean square consistent with rate $O(n^{-1})$ and almost sure consistent with rate $O(\log n/\sqrt{n})$. Also, it is established that $\{\hat{F}_\alpha(u)|0 < u \leq T\}$, $T < \infty$, converges weakly to a Gaussian process whose covariance structure coincides with the limiting covariance structure obtained from Kaplan-Meier's ((1958), J.Amer. Stat. Assoc., 53, 457-481) product limit estimator by Breslow and Crowley ((1974), Ann. Stat. 2, 437-453).